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Technical Report 442

N. P. Smith

Time Optimal Control for a Class of Common Random Disturbances

2 February 1968

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TIME OPTIMAL CONTROL FOR A CLASS OF COMMON RANDOM DISTURBANCES

N. P. SMITH

Group 76

TECHNICAL REPORT 442

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LEXINGTON MASSACHUSETTS

TIME OPTIMAL CONTROL
FOR A CLASS OF COMMON RANDOM DISTURBANCES*

ABSTRACT

This report concerns the time optimal control of a system variable where the controlling input to the system is bounded, as is normally the case in practice. Optimal control is defined here as that control which yields time optimal trajectories. It is shown that time optimal control also yields optimal trajectories in the sense of minimizing the maximum error (if this is the initial error, minimize the overswing next) and the number of oscillations. The problem of optimal control of a second-order system initially in equilibrium and subjected to a large class of commonly occurring random disturbances is solved. Disturbances are considered to be controllable or uncontrollable. The broad class of random disturbances treated herein may have initial nonequilibrium values and consist of a unidirectional uncontrollable portion, followed by a controllable portion of sufficient duration to enable an optimal controller to bring the system to equilibrium. A single control function is derived which suffices to yield optimal trajectories.

Previous attempts which were made to solve similar problems using a statistical approach succeeded only in obtaining approximate optimal control for disturbances restricted to a specific class such as white noise, Brownian motion, etc. In this report, the optimal control applies to a much larger class of random disturbances than previous results, in that it does not restrict the disturbance to be a member of a single statistically defined class such as white noise, etc. The disturbances treated here contain the entire class of disturbances most commonly occurring in practice. This is true because in modern control design the maximum magnitude of the controlling input is sized such that most of the time the disturbances are considered controllable. Thus, the most general class of commonly occurring disturbances of interest to the control designer are those which are uncontrollable for a short duration, followed by long controllable portions.

The success of the technique used in this report lies in assuming that the disturbance is initially known in advance. In this way, a single control function is derived that will yield optimal trajectories for as large a class of disturbances as mathematically possible. This is the best that can theoretically be achieved. Terms contained in the derived control function yield those statistical parameters which are actually required to obtain optimal control. Once the basic form of the necessary statistical parameters has been obtained, the designer can estimate or measure what these parameters are in the actual system for practical optimal control. Experimental results verifying this fact are presented here.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

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* This report is based on a thesis of the same title submitted to the Department of Mechanical Engineering at Purdue University on 27 December 1967 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

CONTENTS

TIME OPTIMAL CONTROL FOR A CLASS OF COMMON RANDOM DISTURBANCES

I. INTRODUCTION

In 1944, Oldenburger[1] derived the time optimal response of a second-order system to step changes in the load and reference value of the controlled variable while trying to optimize aircraft propeller governors. This system was subject to a saturation condition on the manipulated (controlling) variable. In the speed governor field, the saturation condition is always satisfied. Optimal response is obtained by having the servomotor piston speed at maximum or zero at all times.

A paper on optimal nonlinear control by D. MacDonald[2] appeared in 1950, with subsequent papers by Hopkin,[3] Bogner and Kazda,[4] Bushaw,[5] LaSalle,[6] and many others. Hopkin and Wang[7] studied relay systems for random inputs. Rozonoér[8] published a series of papers on the Pontryagin maximum principle concerned with minimizing the duration of trajectories or other functionals, but this class does not include the maximum error. For the system treated in this report, it is shown that by minimizing error response time the maximum error is simultaneously minimized.

In 1953, Flügge-Lotz[9] published a book describing in detail the design and performance of discontinuous control systems using combinations of linear control functions. A paper on optimal control for step and pulse disturbances was written in 1965 by Oldenburger and Chang,[10] who presented the optimal control functions for step and pulse disturbances of arbitrary magnitude and duration. In 1956, Bellman, Glicksberg, and Gross[11] presented a general solution to the problem of optimizing error response time of a system from any given initial state to equilibrium, while the system experienced no external disturbance during this transition. In 1957, Pontryagin, et al.,[12] published a book which introduced the maximum principle. In this work, the general solution for optimizing the error response time was presented, where the controlling variables were subject to a saturation limitation. In 1965, Woodside[13] described an experimental method for finding approximate solutions to problems in optimal control, and also obtained experimental results for a number of examples using this method. The systems were subjected only to a smoothed white noise input and a Brownian motion input. In 1966, Oldenburger[14] published a book concerned with optimizing entire response curves rather than a single index of performance. His treatment of optimal control for arbitrary disturbances involved using a discrete approach and then going through a limiting process to obtain the optimal control function.

In 1966, Kushner[15] wrote a series of papers concerned with the existence and sufficient conditions of optimal stochastic control. He was concerned with minimizing the expected value of a functional. Athans[16] wrote a paper on optimal control with bounded variables, and was concerned with minimizing energy, fuel, and time.

In this report, a system with input ℓ is considered, where ℓ is a disturbance to which the system is subjected; for convenience, this will be called a load disturbance. The system is taken with one output only, this being the controlled variable x. In addition to the disturbance input, there is another quantity, namely, the controlling variable m which is to be varied by the controller to keep x constant or varying with time according to a reference value r; i.e., it is desired to have $r = x$ at all times. Let e denote the error $(r - x)$. In the normal equilibrium state, $r = x = 0$.

For a constant r, many controlled systems may be represented by the equation

$$x' = K_1 m - K_2 \ell \tag{1}$$

where K_1 and K_2 are constants. Let m' denote the rate of change dm/dt of m with respect to time t. In physical problems, m' is bounded so that

$$|m'| \leqslant K_3 \tag{2}$$

for a constant K_3 and the absolute value $|m'|$ of m'. In many problems, except for such factors as lags, m' can be made at any instant to take on any value between $-K_3$ and $+K_3$.

By the substitutions

$$X = \frac{x}{K_1 K_3} \quad , \quad M = \frac{m}{K_3} \quad , \quad u = \frac{m'}{K_3} \quad , \quad L = \frac{K_2 \ell}{K_1 K_3} \tag{3}$$

Relations (1) and (2) become

$$X' = M - L \tag{4}$$

$$|u| \leqslant 1 \quad . \tag{5}$$

Differentiating, Eq. (4) becomes

$$X'' = u - L' \quad . \tag{6}$$

When

$$|L'| < 1 \tag{7}$$

the load disturbance is said to be controllable. The equilibrium state is defined by

$$X = X' = 0 \quad . \tag{8}$$

Now, if the system starts from State (8), but

$$|L'| > 1 \tag{9}$$

it follows that $X'' \neq 0$, and a system error X with $X' \neq 0$ arises. A perfect controller cannot prevent the error, and the disturbance is uncontrollable. If

$$|L'| = 1 \tag{10}$$

in practice, an error will always arise, and one cannot make $(u - L')$ positive or negative as desired to bring the system back to the equilibrium state. Thus, the Cases (9) and (10) are both uncontrollable. A unidirectional disturbance is defined as a disturbance for which the sign of L' is constant. Figure 1 is a typical plot of L' where controllable and uncontrollable portions of the L' vs t-curve alternate. The controllable portions are indicated by C, and the uncontrollable portions by U. In practice, violent increases in L followed by violent decreases will

usually be rare; also, K_3 is chosen so that, for normal disturbances, the uncontrollable portions are followed by long controllable sections (see Ref. 14, pp. 193 and 194).

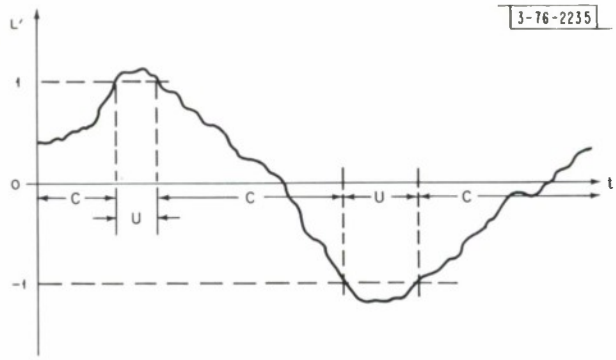

Fig. 1. L' vs time, illustrating typical controllable and uncontrollable intervals.

Suppose that the disturbance L(t) is known not only for the past and present, but also for all future time. It is then possible to derive the optimal control function for this disturbance. This control function can then be utilized for the case when the future of the disturbance is not known. In this way, an optimal control function can be obtained for as large a class of disturbances as mathematically possible. Since no control can be better than that obtained when the disturbance is known in advance, a bound is set on the actual response which can be attained.

Optimal control shall be referred to as that control which yields the minimum duration of the error response when the future of the disturbance L(t) is known arbitrarily far in advance. This report treats the problem of obtaining the control function which yields optimal control in the sense of minimizing the duration of the error response. It is also shown that this control simultaneously minimizes the maximum absolute error $|X_M|$ in the controlled variable X (if this is the initial error, minimize the first overswing next) and the number of oscillations. This report proves that a unique optimal trajectory exists for the system of Eq. (6), with L(t) consisting of a random unidirectional uncontrollable portion followed by a random controllable portion of sufficient duration to enable the optimal controller to bring the system to equilibrium. It is also shown that this optimal trajectory is obtained by having bang-bang control; i.e., the controlling variable u takes on its saturation value

$$|u| = 1 \tag{11}$$

until the system reaches the equilibrium state. The optimal control function Σ is defined as a function which causes u to take on its optimal value (for optimal control) at all times. It is shown that optimal control is obtained by letting

$$u = -\operatorname{sgn} \Sigma \tag{12}$$

where

$$\begin{aligned} u &= -1 \qquad \Sigma > 0 \\ u &= +1 \qquad \Sigma < 0 \\ u &= 0 \qquad \Sigma = 0 \quad . \end{aligned} \tag{13}$$

A phase is defined as a portion of the solution for which u is a constant. In this report, an optimal control function Σ is derived which depends on the instantaneous values of the variables X, X', L, and on the average values of L and L' over a future duration of time. It is also shown that the single control Law (12) suffices to yield optimal trajectories. Figure 2 is a block diagram of the optimal system.

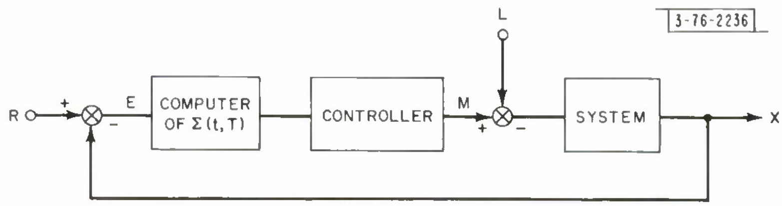

Fig. 2. Block diogrom of optimol control system.

II. THE MAXIMUM PRINCIPLE

The purpose of this section is to use the maximum principle to show the existence of time optimal trajectories to equilibrium, for a system initially at equilibrium and subjected to a unidirectional uncontrollable disturbance followed by a controllable portion of sufficient duration to enable an optimal controller to bring the system to equilibrium. The disturbance may or may not have a step at t = 0. It will be shown here for what general conditions optimal solutions exist, and that optimal control is obtained with a unique two- or one-phase trajectory to equilibrium.

The system described by Eqs. (5) and (6) may be put in the symbolism normally employed for applying the maximum principle. The variables x_1, x_2, x_3, and u shall be used where

$$x_1 = X \quad , \quad x_2 = X' \quad , \quad u = M' \quad , \quad x_3 = t \quad . \tag{14}$$

The system of Eq. (6) becomes

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= u - L' \\ x_3' &= 1 \\ |u| &\leqslant 1 \quad . \end{aligned} \tag{15}$$

Subscript o denotes variables in the initial state; state (x_{10}, x_{20}) is defined as the state of the system at $t = t_o$ so that $(x_1, x_2) = (x_{10}, x_{20})$ at $t = t_o$. The initial conditions at time $t = t_o$ are

$$\begin{aligned} x_1(t_o) &= x_{10} \\ x_2(t_o) &= x_{20} \\ x_3(t_o) &= t_o \\ L(t_o) &= L_o \quad . \end{aligned} \tag{16}$$

Times t_1 and t_2 are the terminal times of the first and second phases, respectively, of a time optimal two- or one-phase trajectory to equilibrium. States (x_{11}, x_{21}) and (x_{12}, x_{22}) are

the states of the system at times $t = t_1$ and $t = t_2$, respectively, so that $(x_1, x_2) = (x_{11}, x_{21})$ at $t = t_1$, and $(x_1, x_2) = (x_{12}, x_{22})$ at $t = t_2$. At the terminal time $t = t_2$, it is required that the system be at the equilibrium state

$$x_1(t_2) = x_{12} = 0$$

$$x_2(t_2) = x_{22} = 0 \quad . \tag{17}$$

Pontryagin's function H is given on p. 60 of Ref. 12 by

$$H = \Psi_o f_o + \Psi_1 f_1 + \Psi_2 f_2 + \Psi_3 f_3 \tag{18}$$

where

$$x'_o = f_o$$

$$x'_1 = f_1 = x_2$$

$$x'_2 = f_2 = (u - L')$$

$$x'_3 = f_3 = 1 \quad . \tag{19}$$

When the disturbance $L(t)$ is known arbitrarily far in advance, it may be expressed in terms of t only. According to p. 59 of Ref. 12, the auxiliary unknown x_3 defined by $x_3 \equiv t$ is used in place of t so that Eqs. (19) may be written in a form not depending explicitly on t. For minimum time, Pontryagin[12] gives $f_o = 1$. Now, Relation (18) becomes

$$H = \Psi_o + x_2\Psi_1 + (u - L')\,\Psi_2 + \Psi_3 \quad . \tag{20}$$

In Ref. 12, it is shown that

$$\Psi_o = \text{constant} \leqslant 0 \quad . \tag{21}$$

Thus, Ψ_o is a nonpositive constant. If $\Psi_2 \neq 0$, it follows that H becomes a maximum with respect to u when

$$u = \operatorname{sgn} \Psi_2 \quad . \tag{22}$$

The adjoint system is given by

$$\Psi'_1 = 0$$

$$\Psi'_2 = -\Psi_1$$

$$\Psi'_3 = \Psi_2 L'' \quad . \tag{23}$$

The solution of Eqs. (23) yields

$$\Psi_1 = k_1$$

$$\Psi_2 = k_2 - k_1 t$$

$$\Psi_3 = k_1 L + L'(k_2 - k_1 t) + k_3 \tag{24}$$

for constants of integration k_1, k_2, and k_3. From Ref. 12, at the terminal time $t = t_2$, the condition

$$\Psi_3(t_2) = 0 \tag{25}$$

must hold to satisfy the transversality condition. By Relations (20) and (24), H becomes

$$H = \Psi_o + k_1(x_2 + L) + k_3 + u(k_2 - k_1 t) \quad . \tag{26}$$

By the second of Relations (24) and Eq. (22), if $(k_1, k_2) \neq (0, 0)$ it follows that u switches once at most from the $u = +1$ to $u = -1$, or vice-versa.

Suppose the system is on a one- or two-phase trajectory to equilibrium with $t_o = 0$. Consider all two- or one-phase trajectories for which $u = -1$ for the first phase until $t = t_1$, and then $u = +1$ for the phase leading to equilibrium at $t = t_2$. For the single-phase case, $t_1 = 0$. Consider first the two-phase case, from which the duration of each phase must be positive, i.e., $t_1 > 0$, and $(t_2 - t_1) > 0$. Later, necessary and sufficient conditions are derived which insure the existence and positive duration of both phases. These are restrictive conditions on the initial state of the system and on the disturbance L(t).

The response of the system of Relations (15) along the first phase $u = -1$ of a two-phase trajectory is obtained by direct integration as

$$x_2 = x_{20} - t - L$$

$$x_1 = x_{10} + x_{20}t - \frac{t^2}{2}\left(1 + \frac{2\bar{L}}{t}\right) \quad , \quad t > 0 \tag{27}$$

where the quantity $t\bar{L}$ is defined by

$$t\bar{L} = \int_o^t L \, dt \quad . \tag{28}$$

Letting $u = -1$ and eliminating x_2 from Eqs. (26) and (27) yields $H = \max H$ along the first phase, where

$$\max H = \Psi_o + k_1 x_{20} - k_2 + k_3 \quad . \tag{29}$$

Since t_1 is the duration of the first phase, the response $x_2(t)$ along the second phase is obtained from the second of Relations (15) and (28) as

$$x_2 = x_{20} - 2t_1 + t - L \quad , \quad t \geqslant t_1 \quad . \tag{30}$$

Along the second phase $u = +1$, Eqs. (26) and (30) yield

$$\max H = \Psi_o + k_1(x_{20} - 2t_1) + k_2 + k_3 \quad . \tag{31}$$

Setting $t = t_2$ in Eqs. (30) with $x_2(t_2) = 0$ yields

$$(t_2 - t_1) = (t_1 - x_{20} + L_2) \quad . \tag{32}$$

Since the duration of the second phase must be positive, it follows from Eq. (32) that a necessary condition for the existence of two-phase optimal trajectories to equilibrium be

$$(t_1 - x_{20} + L_2) > 0 \quad , \quad t_1 \geqslant 0 \tag{33}$$

where $L_2 = L(t_2)$. Later, it will be shown that Relation (33) is always satisfied for the disturbances and initial states treated in this report. Setting $\max H = 0$ in Eqs. (26) and (29), and eliminating t_2 from Eqs. (25) and (32), yield three linear homogeneous equations in k_1, k_2, and k_3:

$$0 = \Psi_o + k_1 x_{20} - k_2 + k_3$$

$$0 = \Psi_o + k_1 (x_{20} - 2t_1) + k_2 + k_3$$

$$0 = k_1 \left[L_2(1 - L_2') + L_2'(x_{20} - 2t_1)\right] + L_2' k_2 + k_3 \tag{34}$$

where $L_2' = L'(t_2)$. By Relation (21), the quantity Ψ_o is a negative constant, or zero. Consider first the case where Ψ_o is a negative constant. Because one of the Ψ's is redundant, we may take

$$\Psi_o = -1 \quad . \tag{35}$$

The solutions of Eqs. (34) and (35) for k_1, k_2, and k_3 are

$$k_1 = \frac{1}{(1 - L_2')(x_{20} - t_1 - L_2)}$$

$$k_2 = \frac{t_1}{(1 - L_2')(x_{20} - t_1 - L_2)}$$

$$k_3 = \frac{-L_2}{(1 - L_2')(x_{20} - t_1 - L_2)} + \frac{-L_2'}{(1 - L_2')} \quad . \tag{36}$$

The disturbance $L(t)$ is assumed to be controllable during the second phase, hence,

$$|L'| < 1 \quad , \quad t_1 \leqslant t \leqslant t_2 \quad . \tag{37}$$

It follows from Relations (32), (36), and (37) that if $(t_2 - t_1) > 0$, and $t_1 > 0$ are satisified, then the conditions

$$k_1 < 0 \quad , \quad k_2 < 0 \tag{38}$$

hold. By Relation (21), other possible optimal trajectories may exist for the case where

$$\Psi_o = 0 \tag{39}$$

instead of $\Psi_o = -1$. There is a solution of $\max H = 0$ for the k's not all vanishing, and hence of the Ψ's not all vanishing identically, only if the determinant of the coefficients in Eqs. (34) is zero. This condition is given by the relation

$$(1 - L_2')(x_{20} - L_2 - t_1) = 0 \quad . \tag{40}$$

By Relations (32) and (37), it follows that Condition (40) is a necessary restrictive condition on x_{20} and $L(t_2)$ for the existence of optimal single-phase trajectories ($u = -1$) to equilibrium. Later, it is shown that Condition (40) does not occur for the initial states and disturbances treated here; thus, it may be disregarded for the optimal control of this report.

By Relations (24) and (38), it follows that

$$\Psi_2 < 0 \quad , \qquad 0 \leqslant t < t_1$$

$$\Psi_2 = 0 \quad , \qquad t = t_1$$

$$\Psi_2 > 0 \quad , \qquad t_1 < t \leqslant t_2 \quad . \tag{41}$$

It has therefore been shown that

$$\Psi_2(t) \neq 0 \quad , \qquad t \neq t_1 \tag{42}$$

for the case of two-phase trajectories to equilibrium, with $u = -1$ followed by $u = +1$. Consider trajectories with a single phase ($u = +1$) to equilibrium. For this case, let $t_1 \to 0$. In the limit, it follows from Relations (24) and (36) that

$$\lim_{t_1 \to 0} \Psi_2 = -k_1 t \quad . \tag{43}$$

Thus, by Relations (38) and (43), the relations

$$\Psi_2 = 0 \quad , \qquad t = 0$$

$$\Psi_2 > 0 \quad , \qquad t > 0 \tag{44}$$

hold for the case of a single phase to equilibrium. The theory for the symmetrical case of one or two phases to equilibrium, with $u = -1$ during the phase leading to equilibrium, is obtained in the same manner as in the preceding argument.

By Relations (22) and (41), H is a maximum with respect to u when u takes on the values

$$u = -1 \quad , \qquad 0 \leqslant t < t_1$$

$$u = +1 \quad , \qquad t_1 < t \leqslant t_2 \quad . \tag{45}$$

It has also been shown by Relation (36) that there are values of k_1, k_2, and k_3 such that $\max H = 0$ for $\Psi_2 \neq 0$ over the duration of the trajectory. It follows from the maximum principle that the control Schedule (45) is a necessary condition for time optimal trajectories to equilibrium. To show the existence of optimal trajectories requires only the proof that the Schedule (45) actually transfers the system from the initial state (x_{10}, x_{20}) to the terminal state (0, 0) while being subjected to the disturbance L(t).

III. EXISTENCE OF TIME OPTIMAL TRAJECTORIES

In this section, it is shown that unique, two-phase time optimal trajectories to equilibrium exist for the case of a unidirectional uncontrollable disturbance followed by a controllable portion of sufficient duration to enable an optimal controller to bring the system to equilibrium. It is also shown that these time optimal trajectories are also optimal in other engineering senses such as that the maximum error is minimized and there is no overshoot.

Consider the system of Eqs. (15) with the initial states

$$x_{10} = 0$$

$$x_{20} = -L_o \geqslant 0 \tag{46}$$

and disturbance L^* consisting of a unidirectional uncontrollable portion which terminates at $t = t_u$. For $t > t_u$, the disturbance L^* is controllable until the system reaches the equilibrium state at $t = t_2$. We have thus defined a class of disturbances L^* satisfying the conditions

$$|L^{*\prime}| \geqslant 1 \quad , \quad 0 < t \leqslant t_u$$
$$|L^{*\prime}| < 1 \quad , \quad t_u < t \leqslant t_2 \quad . \tag{47}$$

First, we treat the case where

$$L^{*\prime} < -1 \tag{48}$$

for the uncontrollable interval $0 < t \leqslant t_u$, as shown in Fig. 3. In order for the system to reach equilibrium along a time optimal two-phase trajectory, we must have $u = -1$ for the first phase. Hence, the x_1-curve will bend away from the t-axis a minimum. This control will result in a minimum error x_1 for each instant t in the interval $(0, t_u)$. From Eqs. (15) and the fact that $L^{*\prime} < -1$ during the interval $0 < t \leqslant t_u$, it follows that

$$x_1 > 0$$
$$x_2 > 0 \quad , \quad 0 < t \leqslant t_u$$

as shown in Figs. 3 and 4. For $t > t_u$, the disturbance L^* becomes controllable. It follows by Eqs. (15) that if $u = -1$ holds for $t > t_u$, at some time $t = t_M$ where $t_M > t_u$, we have

$$x_1(t_M) > 0$$
$$x_2(t_M) = 0 \tag{49}$$

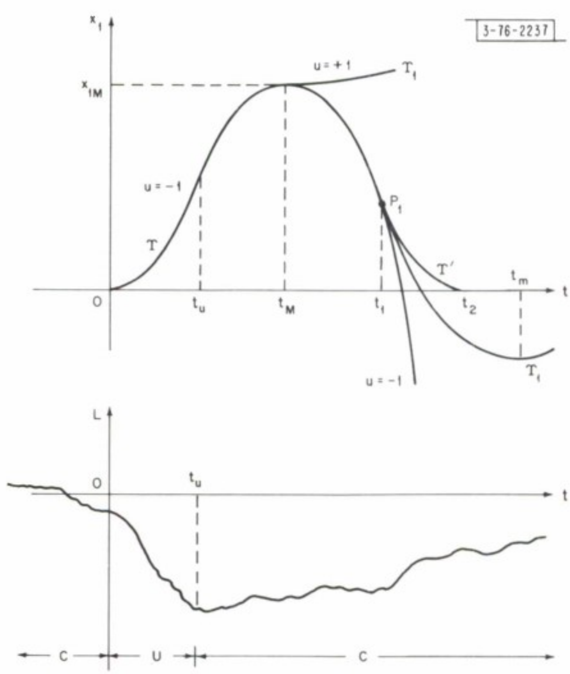

Fig. 3. System response to uncontrollable disturbance followed by controllable disturbance.

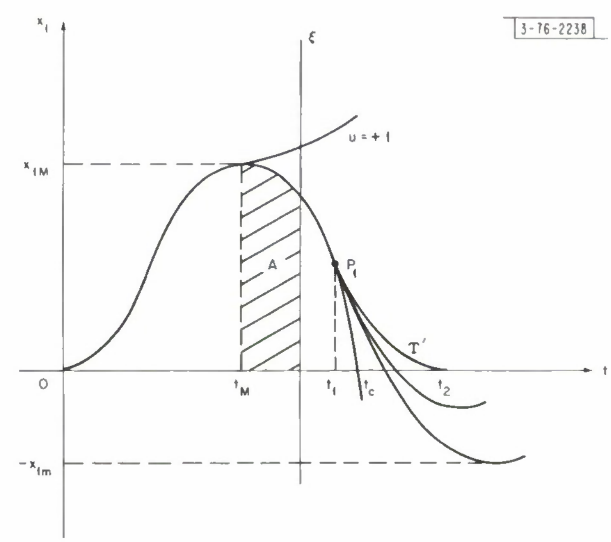

Fig. 4. System trajectories for various switching times.

and x_1 reaches the maximum value of x_{1M} at time $t = t_M$. It follows from Eqs. (15) that

$$x_1 > 0$$

$$x_2 > 0 \quad , \quad 0 < t < t_M \tag{50}$$

as shown in Figs. 3 and 4.

If $u = -1$ holds for $t > t_M$, the inequality $|L^{*\prime}| < 1$ implies that $x_2' < 0$. We assume that there is a number μ and an instant t_4 such that for $t > t_4$, where $t_4 > t_u$, we have

$$|L^{*\prime}| < \mu < 1 \quad . \tag{51}$$

It follows that with $u = -1$, the x_1-curve will eventually reach the t-axis. Let t_c denote the time at which the x_1-curve crosses the t-axis, as shown in Fig. 4. For any vertical line ξ between t_M and t_c, the area between the x_1-curve, the t-axis, the line $t = t_M$, and the line ξ is a minimum. At the crossing point $t = t_c$, we have

$$x_1(t_c) = 0$$

$$x_2(t_c) < 0 \quad . \tag{52}$$

The curve from $t = t_M$ to $t = t_c$ is not optimal in the sense that this results in an overswing for $t > t_c$ with $x_1 = x_{1m}$ at the minimum point for $x_{1m} < 0$. To minimize $-x_{1m}$, we must have $u = +1$ for $t > t_c$. The value of $-x_{1m}$ can be diminished further by letting u switch from -1 to $+1$ at a point P_1 and time $t = t_1$, where $t_1 < t_c$. As P_1 moves to the left in Fig. 3, the minimum point x_{1m} rises. If we take P_1 at the maximum point $t = t_M$, the x_1-curve will rise to the right of t_M as shown in Figs. 3 and 4. It follows that there is a time $t = t_1$ between t_M and t_c such that

$$t_M < t_1 < t_c \tag{53}$$

where the x_1-curve to the right of P_1 touches the t-axis for a value $t = t_2$. This point is the equilibrium state

$$x_1(t_2) = x_2(t_2) = 0 \quad .$$

Starting at $t = t_1$, the arc for which $u = +1$ up to the equilibrium state at $t = t_2$ is denoted by T' in Figs. 3 and 4. The curve in Fig. 3, composed of the x_1-curve from $t = 0$ to $t = t_1$ (for which $u = -1$) and the T'-curve from $t = t_1$ to $t = t_2$ (for which $u = +1$), is called the T-curve and is optimal, as shown by the following argument.

Every other solution T_1 passes above or coincides with the T-curve from $t = 0$ to $t = t_1$, or is identical with it up to a value t_3 of t, where $t_1 \leqslant t_3 < t_2$, after which it crosses the t-axis and attains a minimum point below the t-axis at an instant $t = t_m$ where $t_m > t_2$, as shown in Fig. 3. If $T_1 = T$ for $0 \leqslant t \leqslant t_1$, then for $t > t_1$ any curve T_1 will lie on or between the curve $u = -1$ and T'. In fact, for any t where $t > t_1$

$$-(1 + L^{*\prime}) \leqslant x_2' \leqslant (1 - L^{*\prime}) \quad .$$

The slope x_2 of the curve with $u = -1$ decreases at the maximum rate, whereas, for T to the right of $t = t_1$, it increases at the maximum rate.

The T-curve is unique and optimal in that all other solutions yield overswings and greater durations from t_1 to equilibrium. We have shown that the maximum error x_1 is minimized by a unique two-phase trajectory to equilibrium with a phase $u = -1$ to $t = t_1$ followed by a phase $u = +1$ to $t = t_2$, where $(x_1, x_2) = (0, 0)$ at $t = t_2$. It follows that the conditions

$$
\begin{aligned}
& x_1 > 0 \\
& x_2 < 0 \quad , \quad t_M < t < t_2
\end{aligned}
\tag{54}
$$

also hold along the optimal trajectory. A similar argument holds for the case where $x_{10} = 0$, $x_{20} \geqslant 0$, and $L^{*\prime} > +1$ for the uncontrollable interval $0 < t \leqslant t_u$, where $t_u < t_1$.

By the preceding argument, we have proved the existence of unique two-phase trajectories to equilibrium with $t_1 > 0$, and $(t_2 - t_1) > 0$ for the system of Eqs. (15) subjected to the class of disturbance L^*. It follows that a unique time optimal solution exists for which Relations (38) and (41) hold and the control Schedule (45) yields $\max H = 0$ with $\Psi_2 \neq 0$ along the T-curve.

IV. OPTIMAL RESPONSE OF SYSTEM

Equations are now derived which describe the optimal response of the system of Eqs. (15) when subjected to a disturbance L(t). General restrictive conditions on L(t) are given which must be satisfied in order for the system to reach equilibrium along an optimal two- or one-phase trajectory. It is verified here that the class of disturbances L^* treated in this report do satisfy these conditions.

Let the quantities $\overline{L}'_{ij}$ and $\overline{\alpha}_{ij}$ be defined by

$$
\begin{aligned}
\overline{L}'_{ij} &= \frac{1}{(t_j - t_i)} \int_{t_i}^{t_j} L' \, dt \quad , \quad i = 0, 1; \; j = 1, 2 \\
\overline{\alpha}_{ij} &= \frac{2}{(t_j - t_i)^2} \int_{t_i}^{t_j} (t_j - t) \, L' \, dt \quad .
\end{aligned}
\tag{55}
$$

By direct integration of Eqs. (15) with $u = -1$ over the interval from $t_i = 0$ to $t_j = t_1$, the state of the system at $t = t_1$ becomes

$$
\begin{aligned}
x_{21} &= x_{20} - t_1 (1 + \overline{L}'_{01}) \\
x_{11} &= x_{10} + x_{20} t_1 - \frac{1}{2} t_1^2 (1 + \overline{\alpha}_{01})
\end{aligned}
\tag{56}
$$

where the initial value of x_{20} arises from an initial value of L at $t = 0$,

$$
x_{20} = -L_o \quad . \tag{57}
$$

Let (x_{11}, x_{21}) be the initial state of the system at the start of the second phase. The state of the system at the terminal time $t = t_2$ is also obtained by direct integration of Eqs. (15) with $u = +1$ over the interval from $t_i = t_1$ to $t_j = t_2$, thus,

$$
\begin{aligned}
x_{22} &= x_{21} + (t_2 - t_1)(1 - \overline{L}'_{12}) \\
x_{12} &= x_{11} + (t_2 - t_1) x_{21} + \frac{1}{2} (t_2 - t_1)^2 (1 - \overline{\alpha}_{12}) \quad .
\end{aligned}
\tag{58}
$$

In order for the system to reach equilibrium at $t = t_2$, along a phase $u = +1$ from the state (x_{21}, x_{11}), it follows from Eqs. (17) and (58) that

$$x_{21} = -(t_2 - t_1)(1 - \bar{L}'_{12})$$
$$x_{11} = \frac{1}{2}(t_2 - t_1)^2 (1 - 2\bar{L}'_{12} + \bar{\alpha}_{12}) \quad . \qquad (59)$$

We consider only disturbances which are controllable along the second phase; then, by Relation (37), it follows that

$$|L(t) - L(t_1)| < (t - t_1) \quad , \quad t_1 < t \leqslant t_2 \quad . \qquad (60)$$

Hence, it follows from Relation (60) and the first of Relations (55) that

$$|\bar{L}'_{12}| < 1 \quad . \qquad (61)$$

In view of Relations (55) and (37), we may also write

$$(\bar{\alpha}_{12} - 2\bar{L}'_{12}) \leqslant \frac{2}{(t_2 - t_1)^2} \int_{t_1}^{t_2} |L(t) - L(t_2)|\, dt \quad . \qquad (62)$$

By Relation (37), it also follows that

$$|L(t) - L(t_2)| < (t_2 - t) \quad , \quad t_1 < t \leqslant t_2 \quad . \qquad (63)$$

Thus, by Relations (62) and (63), we have

$$|\bar{\alpha}_{12} - 2\bar{L}'_{12}| < 1 \quad . \qquad (64)$$

Since $(t_2 - t_1) > 0$, it follows from Relations (59), (61), and (64) that the state of the system at $t = t_1$ must satisfy the conditions

$$x_{21} < 0$$
$$x_{11} > 0 \quad . \qquad (65)$$

By Relations (56), we may express Conditions (65) in the form

$$x_{20} - t_1(1 + \bar{L}'_{01}) < 0$$
$$x_{10} + x_{21}t_1 + \frac{1}{2} t_1^2(1 + 2\bar{L}'_{01} - \bar{\alpha}_{01}) > 0 \quad . \qquad (66)$$

By Relations (46) and since $t_1 > 0$, it follows that the conditions

$$(1 + \bar{L}'_{01}) > 0$$
$$(1 + 2\bar{L}'_{01} - \bar{\alpha}_{01}) > 0 \qquad (67)$$

must be satisfied in order for Conditions (65) to be satisfied. We shall now show that Conditions (67) are satisfied for the class of disturbances L^* treated in this report.

By Eqs. (27) and (49), the state of the system at $t = t_M$ is

$$x_2(t_M) = 0 = x_{20} - t_M(1 + \bar{L}'_{0M})$$

$$x_1(t_M) = x_{20}t_M - \frac{1}{2}(1 + \bar{\alpha}_{0M})\, t_M^2 \tag{68}$$

where

$$\bar{L}'_{0M} = \frac{1}{t_M}\int_0^{t_M} L^{*\prime}\, dt$$

$$\bar{\alpha}_{0M} = \frac{2}{t_M^2}\int_0^{t_M} (t_M - t)\, L^{*\prime}\, dt \quad . \tag{69}$$

Eliminating x_{20} from Relations (68) yields

$$x_{1M} = \frac{1}{2}\, t_M^2\,(1 + 2\bar{L}'_{0M} - \bar{\alpha}_{0M}) \quad . \tag{70}$$

Since $x_{1M} > 0$ and $x_{20} \geqslant 0$, it follows from the first of Relations (68) and Relation (70) that

$$(1 + \bar{L}'_{0M}) \geqslant 0$$

$$(1 + 2\bar{L}'_{0M} - \bar{\alpha}_{0M}) > 0 \quad . \tag{71}$$

We define the quantities $\bar{L}'_{M1}$ and $\bar{\alpha}_{M1}$ to be

$$\bar{L}'_{M1} = \frac{1}{(t_1 - t)}\int_{t_M}^{t_1} L^{*\prime}\, dt$$

$$\bar{\alpha}_{M1} = \frac{2}{(t_1 - t_M)^2}\int_{t_M}^{t_1} (t_1 - t)\, L^{*\prime}\, dt \quad . \tag{72}$$

By Relations (55), (69), and (72), we have

$$\bar{L}'_{01} = \frac{t_M}{t_1}\,\bar{L}'_{0M} + \left(1 - \frac{t_M}{t_1}\right)\bar{L}'_{M1}$$

$$\bar{\alpha}_{01} = \frac{1}{t_1^2}\,[t_M^2(\bar{\alpha}_{0M} - 2\bar{L}'_{0M}) + \bar{\alpha}_{M1}(t_1 - t_M)^2 + 2\bar{L}'_{0M}t_M t_1] \quad . \tag{73}$$

Since L^* is controllable over the interval $t_M \leqslant t \leqslant t_1$, we have

$$|\bar{L}'_{M1}| < 1 \quad , \quad |\bar{\alpha}_{M1}| < 1 \quad . \tag{74}$$

Hence, by Relations (71), (73), and (74), it follows that Conditions (67) are satisfied for the class of disturbances L^* and initial conditions of Relations (46) treated in this report. A similar argument holds for the case where $x_{10} = 0$, $x_{20} \geqslant 0$, and $L^{*\prime} > +1$ for the uncontrollable interval $t \leqslant t_u$.

V. OPTIMAL CONTROL FUNCTION

We now introduce a control function $\Sigma(t, T)$ and prove that time optimal control for the class of disturbances L^* and initial states of Relations (46) is obtained by the control Law (12).

Let T be the duration of the second phase $u = +1$, where

$$T = (t_2 - t_1)$$

Let $\overline{L}'$ and $\overline{L}$ be defined by the relations

$$\overline{L}' = \frac{1}{T} \int_t^{t+T} L^{*\prime} \, dt$$

$$\overline{L} = \frac{1}{T} \int_t^{t+T} L^{*} \, dt \quad . \tag{75}$$

We introduce the function $\Sigma(t, T)$ where

$$\Sigma(t, T) = x_1(1 - \overline{L}' \operatorname{sgn} x_1)^2 + \frac{1}{2} x_2 \, |x_2| \cdot \left| 1 + 2(\overline{L}' + \frac{L - \overline{L}}{T}) \operatorname{sgn} x_2 \right| \quad . \tag{76}$$

Eliminating $(t_2 - t_1)$ from Relations (59) yields $\Sigma_1 = 0$, where

$$\Sigma_1 = \Sigma(t_1, T) \quad . \tag{77}$$

Note that Eq. (76) also yields $\Sigma_1 = 0$ for a pre-equilibrium phase $u = -1$ as well as for $u = +1$, where, again, $t = t_1$ at the start and $t = t_2$ at the end of this phase.

We shall now prove that time optimal control is obtained by the control law

$$u = -\operatorname{sgn} \Sigma(t, T) \tag{78}$$

where

$$u = -1 \quad , \quad \Sigma(t, T) > 0$$

$$u = +1 \quad , \quad \Sigma(t, T) < 0$$

$$u = 0 \quad , \quad \Sigma(t, T) = 0 \quad . \tag{79}$$

By the maximum principle, we have shown that time optimal control is obtained by the control Schedule (45). From Relations (46) and Eq. (76), it follows that

$$\Sigma(0, T) \geqslant 0 \quad . \tag{80}$$

First, consider the case where $\Sigma(0, T) = 0$. According to Relations (76), this can happen only if the relation

$$x_2 \, |x_2| \cdot \left| 1 + 2(L' + \frac{L - \overline{L}}{T}) \operatorname{sgn} x_2 \right| = 0 \tag{81}$$

is satisfied at $t = 0$. If Relation (81) is satisfied, we have $u = 0$ at $t = 0$ by Relations (79). It follows by Relation (48) and the second of Relations (15) that

$$x_2' > 0$$

$$x_2 > 0 \quad , \quad t = +\delta$$

$$x_1 > 0 \tag{82}$$

where δ is a small positive quantity. Thus, from Relations (76) and (82), we have

$$\Sigma(\delta, T) > 0 \quad . \tag{83}$$

By Relations (50), (76), and (83), it follows that

$$\Sigma(t, T) > 0 \quad , \quad \delta \leqslant t \leqslant t_M \quad . \tag{84}$$

Thus, the control Law (78) yields optimal control for the interval $0 < t \leqslant t_M$. The proof of the relation

$$\Sigma(t, T) > 0 \quad , \quad t_M < t < t_1 \tag{85}$$

follows from p. 212 of Ref. 14. For a given t, the functions $\Sigma(t, T)$ belong to a one-parameter class with T as the parameter. We have shown that, until $t = t_M$ on an optimal trajectory, any member $\Sigma(t, T)$ of this class will do in Relation (78); in fact, any member will suffice until we reach a pair of values (t_1, T_1) of (t, T) for which

$$\Sigma(t_1, T_1) = 0 \tag{86}$$

holds. The function $\Sigma(t, T)$ will change sign at a time $t = t_1$ where $t_M < t_1 < t_e$, as shown from the following argument. Note in Fig. 3 that if we remain on the $u = -1$ curve, at some time $t = t_e$ we have $x_1(t_e) = 0$, and $x_2(t_e) < 0$; thus, $\Sigma(t_e, T) < 0$ for any T. It follows that for any T there is a point between $t = t_M$ and $t = t_e$ where the function $\Sigma(t, T)$ changes sign.

By Relations (54), we have

$$x_1 > 0 \quad , \quad t_1 \leqslant t < t_2 \quad . \tag{87}$$

We have shown that a unique two-phase time optimal trajectory to equilibrium is obtained with $u = -1$ along the first phase until $\Sigma(t, T) = 0$, followed by $u = +1$ along the second phase until $t = t_2$. It follows that the switch at $t = t_1$ occurs when $\Sigma(t_1, T) = 0$. Thus, the $\mathcal{T}$-curve in Fig. 3 is optimal where t_1 is the smallest value of t for which there exists a value of t_2 such that $\Sigma(t, T) = 0$ at $t = t_1$, and $(x_1, x_2) = (0, 0)$ at $t = t_2$. A similar argument holds for the case where $x_{10} = 0$, $x_{20} \geqslant 0$, and $L^{*\prime} \geqslant 1$ for the uncontrollable interval $0 < t \leqslant t_u$, where $t_u < t_1$. We have proved the following theorem.

Theorem 1.

Let a second-order system [Eqs. (15)] be at equilibrium for $t \leqslant 0$, and let the system be subjected to a disturbance $L(t)$ for $t > 0$, where $L(t)$ is made up of an initial uncontrollable portion followed by a controllable portion sufficiently long for an optimal controller to bring the system to equilibrium. Let $\overline{L}$ and $\overline{L'}$ be the averages

$$\frac{1}{T}\int_t^{t+T} L \, dt \quad , \quad \frac{1}{T}\int_t^{t+T} L' \, dt \tag{88}$$

of L and L' for the time interval $(t, t + T)$, and let the control function $\Sigma(t, T)$ be defined by Eq. (76). Optimal response in the sense of minimum time, maximum error minimized, and no overshoot is obtained with the control law

$$u = -\operatorname{sgn} \Sigma(t, T) \qquad \text{[Law (78)]}$$

until a pair of values (t_1, T_1) of (t, T) are attained for which

$$\Sigma(t_1, T_1) = 0 \quad . \tag{89}$$

At the instant $t = t_1$, control is switched to the equation

$$u = \operatorname{sgn} x_1$$

and remains at this value of u until equilibrium is attained.

A similar argument holds for the initial states $x_{10} = 0$, $x_{20} < 0$, with $L' > +1$ for the uncontrollable portion, or $x_{10} = 0$, $x_{20} < 0$, with $L' < -1$ for the uncontrollable portion. Thus, the results of Theorem 1 also apply for these conditions.

VI. DELAY IN SWITCHING

In this section, we consider the effect of a time delay in switching u from $+1$ to -1, or vice-versa, for the optimal control of this report. It is shown that, for a time delay in switching u, the control Law (78) with the control function of Eq. (76) yields near-optimal trajectories to equilibrium. As the delay in switching becomes small, the near-optimal approaches the optimal trajectory. It is also shown here that a single control function of Eq. (76) suffices to yield optimal trajectories to equilibrium.

The original idea upon which the following argument is based is credited to Oldenburger.[1] Consider the case where $u = -1$ for the first phase of an optimal trajectory until for values t_1 of t and T_1 of T we have $\Sigma(t_1, T_1) = 0$. At this point, u switches from -1 to $+1$ for optimal control. In practice, there will be a delay in switching. Suppose that we switch at $t = (t_1 + \epsilon)$, where $\epsilon > 0$, instead of at $t = t_1$ for which control Law (78) holds. Thus, at $t = t_1$ we leave the optimal trajectory where $u = +1$ after $t = t_1$, and remain on $u = -1$ until $t = (t_1 + \epsilon)$. This trajectory is non-optimal. Since $u = -1$ on the non-optimal trajectory for each t in the interval $(t_1, t_1 + \epsilon)$, it follows that at each instant t in this interval both x_1 and x_2 are less than they would be if we had stayed on the optimal trajectory ($u = +1$). If ϵ is small enough, the quantities $(\operatorname{sgn} x_2)$, L, $\overline{L}$, and $\overline{L}'$ occurring in $\Sigma(t, T_1 - t + t_1)$ are the same at $t = (t_1 + \epsilon)$ for both the optimal and non-optimal trajectories. Since x_1 and x_2 are less for the non-optimal than for the optimal trajectories, it follows that $\Sigma(t, T_1 - t + t_1)$ is less at $t = (t_1 + \epsilon)$ for the non-optimal than for the optimal trajectory. Taking the time derivative $\Sigma'(t, T)$ of $\Sigma(t, T)$ in Eq. (76) and evaluating for the interval $t_1 \leqslant t \leqslant t_2$ yields

$$\Sigma'(t, T) = 0 \quad , \quad t_1 \leqslant t \leqslant t_2 \quad . \tag{90}$$

Since $\Sigma(t_1, T_1) = 0$, by Eq. (90) it follows that

$$\Sigma(t, T) = 0 \quad , \quad t_1 \leqslant t \leqslant t_2 \quad . \tag{91}$$

Since Relation (91) holds along the phase $u = +1$ to equilibrium for the optimal case, it follows that $\Sigma(t, T_1 - t + t_1) < 0$ at $t = (t_1 + \epsilon)$ on the non-optimal trajectory. If we use control Law (78) with the control function $\Sigma(t, T_1 - t + t_1)$, it follows that we switch to $u = +1$ at $t = (t_1 + \epsilon)$. For the interval $(t_1 + \epsilon) \leqslant t \leqslant t_2$, the sign of $\Sigma(t, T_1 - t + t_1)$ remains unchanged for both the optimal and non-optimal trajectories except that x_1 and x_2 are less for the non-optimal than for the optimal trajectory. For both trajectories, we have $u = +1$ over the time interval $(t_1 + \epsilon, t_1 + T_1)$, but x_1 and x_2 for the non-optimal trajectory are both less than for the optimal until $x_1 = 0$ on the

optimal trajectory; then, $\Sigma(t,\ T_1 - t + t_1) < 0$ for the non-optimal trajectory until $x_1 = 0$. Thus, for the non-optimal trajectory, the sign of u is opposite that of $\Sigma(t,\ T_1 - t + t_1)$ over the interval $(t_1 + \epsilon) \leqslant t \leqslant (t_1 + T_1)$.

On the non-optimal trajectory, $(x_1, x_2) \neq (0, 0)$ at $t = (t_1 + T_1)$; in fact, $x_1 < 0$ at this t, from which x_1 becomes zero before $t = (t_1 + T_1)$. Let the time t at which x_1 becomes zero be denoted by $t = (t_1 + T_1 - \Delta)$, where $\Delta > 0$. When the point at $t = (t_1 + T_1 - \Delta)$ on the non-optimal trajectory described above is attained, one may start over again with a function $\Sigma(t, T)$ for arbitrary T and repeat the argument. This process can be continued so that the control Law (78) always applies for appropriate choices of $\Sigma(t, T)$. As ϵ goes to zero, the non-optimal trajectory approaches the optimal. We have proved the following theorem.

Theorem 2.

> For the system and disturbance of Theorem 1, optimal control is attained in practice by letting
>
> $$u = -\operatorname{sgn} \Sigma(t, T) \qquad \text{[Law (78)]}$$
>
> where T in $\Sigma(t, T)$ is arbitrary until for a pair of values (t_1, T_1) of (t, T)
>
> $$\Sigma(t_1, T_1) = 0 \quad .$$
>
> Thereafter, control Law (78) holds with $T = (T_1 - t + t_1)$ until $x_1 = 0$. At this point, a new function $\Sigma(t, T)$ is used with arbitrary T and the process is repeated so that control Law (78) always applies for appropriate choices of T in $\Sigma(t, T)$. Thus, the single control Law (78) suffices to yield optimal trajectories to equilibrium.

VII. EQUIVALENCE OF REFERENCE INPUT TO LOAD CHANGES

In the preceding theorems, we considered the load disturbance to be the only input to the system. Now, we shall consider the system of Eqs. (15) subjected to two inputs: namely, a reference input r(t), and the load disturbance L(t). We shall show that, in effect, the addition of a reference input is equivalent to replacing the disturbance L(t) in Eqs. (15) by a new disturbance $L_r(t)$ with no reference input. Thus, the preceding theorems will also apply to the system of Eqs. (15) subjected to both a load disturbance L and a reference input r.

Let e be the difference between the reference value r of the controlled variable x_1 given by

$$e = x_1 - r \quad . \tag{92}$$

The system of Eqs. (15) is now replaced by

$$e'' = u - L_r' \quad , \qquad |u| \leqslant 1 \tag{93}$$

where

$$L_r = L + r' \quad . \tag{94}$$

It follows that a variation in the reference r(t) is equivalent to a variation r'(t) in the load L_r. The variables x_1 and x_2 are replaced by the error e and the time derivative of the error e',

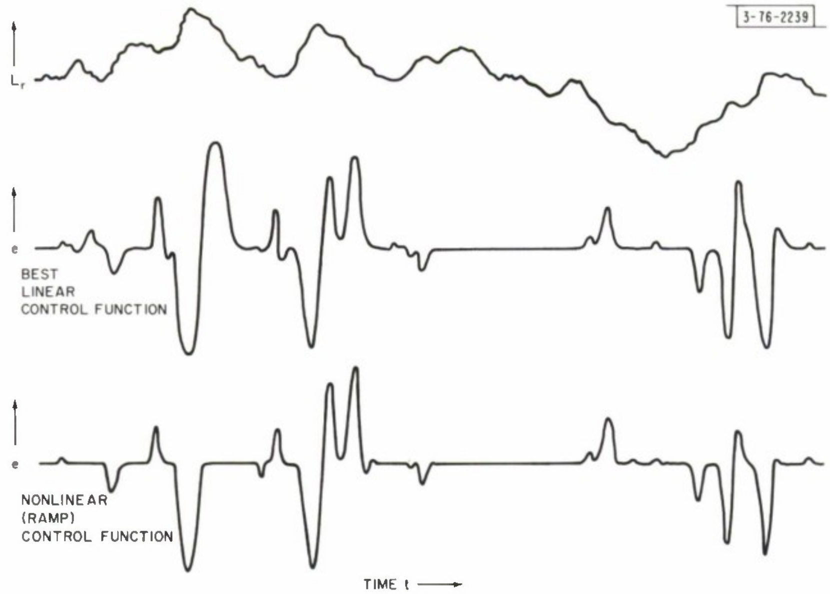

Fig. 5. Experimental results of system response to random disturbance using linear and nonlinear control functions.

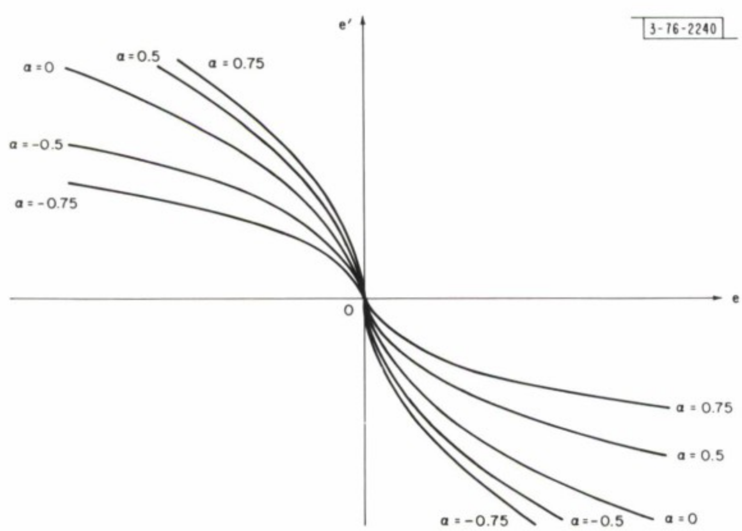

Fig. 6. Phase plane trajectories of $\Sigma = 0$ for $L_r' =$ constant $= a$.

respectively. Since the average of the sum of two variables is equal to the sum of the averages, we may write the average of L_r and L'_r over the interval of time T as

$$\overline{L}_r = \overline{L} + \overline{r}'$$

$$\overline{L'}_r = \overline{L}' + \overline{r}'' \quad . \tag{95}$$

From Eqs. (76) and (95), we have the optimal control function $\Sigma(t, T)$ for the equivalent load L_r given by

$$\Sigma(t, T) = e(1 - \overline{L}'_r \operatorname{sgn} e)^2 + \frac{1}{2} e' |e'| \cdot \left| 1 + 2\left(L'_r + \frac{L_r - \overline{L}_r}{T}\right) \operatorname{sgn} e' \right| \quad . \tag{96}$$

For L_r to be controllable by Relation (7), we must have

$$|L' + r''| < 1 \quad . \tag{97}$$

We have shown the following statement to be true. Let the system of Eqs. (15) be in equilibrium for $t \leqslant 0$ and be subjected to two inputs $L(t)$ and $r(t)$ for $t > 0$, where

$$L_r(t) = L(t) + r'(t) \tag{98}$$

and L_r is made up of an initial uncontrollable portion followed by a controllable portion sufficiently long for an optimal controller to bring the system to equilibrium. According to Theorems 1 and 2, optimal response is obtained with the control Law (78) and the control function given in Eq. (96).

Consider the class of inputs $r(t)$ which may be closely approximated by a quadratic function of time, i.e.,

$$r \cong r_o + r_1 t + r_2 t^2 \tag{99}$$

for constants r_o, r_1, and r_2. Consider also the class of disturbances which may be closely approximated by a ramp where

$$L \cong L_o + at \tag{100}$$

for constant a. By Relation (94), the equivalent disturbance L_r is a ramp. The quantity r is an arbitrary value for $t < 0$ and there may be a step in L_r at $t = 0$. By Relations (99) and (100), the optimal control function $\Sigma(t, T)$ of Eq. (96) may be approximated closely by

$$\Sigma(t, T) \cong e\,|1 - L'_r \operatorname{sgn} e| + \frac{1}{2} e' |e'| \quad . \tag{101}$$

We use control Law (78) with $\Sigma(t, T)$ approximated by Relation (101) for suboptimal control. When $|L'_r| \ll 1$, the optimal control function

$$\Sigma = e + \frac{1}{2} |e'| e' \tag{102}$$

for step changes is effective in obtaining suboptimal response. Experimental results were obtained using the control function of Relation (101) with the control Law (78). This suboptimal control was effective in obtaining suboptimal response, as shown in Fig. 5.

The curves $\Sigma = 0$ for Σ of Relation (101) for various values of L'_r, where L'_r = constant = a, are shown in Fig. 6. In practice, one may not wish or be able to measure the quantity L'_r, in

which case it is necessary to express the control function Σ of Relation (101) in terms of e, e', and e'' only. When u = 1, Relations (93) yield

$$|L_r'| = |1 - |e''||$$

$$\operatorname{sgn} L_r' = \operatorname{sgn}(e'' - e''^3) \qquad (103)$$

by which we may express L_r' in the form

$$L_r' = \Big|1 - |e''|\Big| \cdot \operatorname{sgn}(e'' - e''^3) \quad . \qquad (104)$$

We may now substitute L_r' from Eq. (104) into Σ of Eq. (101) and obtain a control function in terms of e, e', and e'' only. Because of ever-present noise, some filtering of e'' is necessary in order to employ Σ of Eq. (101) with L_r' of Eq. (104). Normally, the system of Relations (93) will not describe the system exactly. If the approximation is good, the control will be suboptimal.

Laboratory tests of the use of the control function Σ of Eq. (101) gave substantial improvement for random disturbances over what could be obtained by known techniques. Here, L_r' is now allowed to be a variable. To explain the remarkable improvement obtained in the laboratory, we considered optimal control for more general disturbances than ramps.

VIII. EXPERIMENTAL RESULTS

We now discuss tests which were run to evaluate system performance for random disturbances using the control Law (78) with the control function of Relation (101) which is optimal for ramp disturbances. Thus, if the disturbance may be closely approximated with ramps, the control function of Relation (101) will be a good approximation to the optimal control function of Eq. (96) and will yield suboptimal response.

The system of Eqs. (15) was simulated on an analog computer in the control systems laboratory. The disturbance $L_r(t)$ was generated with a random signal generator and filtered twice to yield controllable and uncontrollable intervals similar to those shown in Fig. 1. Control Law (78) with the control function of Relation (101) were simulated along with the system to yield closed-loop control. To evaluate system performance, a second system was simulated identical to the first except for a linear control function in place of Σ of Eq. (101). The responses of the two systems were compared over long duration runs, and a typical section of these runs is shown in Fig. 5. The linear control function used in the second system was of the form

$$\Sigma = e - \alpha e' + \beta e'' \qquad (105)$$

for constants α and β. If these constants are properly chosen, the linear function of Eq. (105) will give reasonably close response to that obtainable with Σ of Eq. (101) for a given disturbance. See Fig. 5 where the coefficients in Eq. (105) were chosen to yield a minimum rms value of the controlled variable e for the random disturbance L_r shown. Many of the same type tests shown in Fig. 5 were run and, in all cases where the disturbance was composed of short uncontrollable sections followed by long controllable sections, the control function of Relation (101) yielded better response than the linear function of Eq. (105), i.e., shorter error response duration, smaller maximum error, and fewer overshoots. It was also determined by tests with Σ of Eq. (101) with the $\overline{L_r'}$ term omitted that not nearly as good results can be expected for this case. Thus, a curve $L_r(t)$ is better approximated by a broken line formed by secants than by using a staircase approximation.

IX. SUMMARY AND CONCLUSIONS

The problem of time optimal control of a second-order system initially at equilibrium and subjected to a class of common random disturbances is solved. The class of disturbances treated are those most commonly encountered in practice, and are made up of an initial uncontrollable portion followed by a controllable portion of sufficient duration to enable an optimal controller to bring the system to equilibrium. Necessary conditions for optimal control were derived using the maximum principle, and these conditions were shown to be satisfied with the control of this report. Necessary conditions for the existence of optimal trajectories to equilibrium were also derived. These are restrictive conditions on the initial state of the system and disturbance, and were also shown to be satisfied for the class of disturbances and initial states treated here.

A control law and control function were derived to yield optimal trajectories to equilibrium, and these trajectories were shown to be also optimal in other engineering senses such as that the maximum error was minimized and there was no overshoot. It was shown that a single control law with a single control function sufficed to yield optimal trajectories to equilibrium.

Laboratory tests were made comparing an approximate optimal control function with the best linear control function. These tests were run with the system subjected to a random disturbance, and the results indicated superior control over what could be obtained by known techniques.

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13. ABSTRACT

This report concerns the time optimal control of a system variable where the controlling input to the system is bounded, as is normally the case in practice. Optimal control is defined here as that control which yields time optimal trajectories. It is shown that time optimal control also yields optimal trajectories in the sense of minimizing the maximum error (if this is the initial error, minimize the overswing next) and the number of oscillations. The problem of optimal control of a second-order system initially in equilibrium and subjected to a large class of commonly occurring random disturbances is solved. Disturbances are considered to be controllable or uncontrollable. The broad class of random disturbances treated herein may have initial nonequilibrium values and consist of a unidirectional uncontrollable portion, followed by a controllable portion of sufficient duration to enable an optimal controller to bring the system to equilibrium. A single control function is derived which suffices to yield optimal trajectories.

Previous attempts which were made to solve similar problems using a statistical approach succeeded only in obtaining approximate optimal control for disturbances restricted to a specific class such as white noise, Brownian motion, etc. In this report, the optimal control applies to a much larger class of random disturbances than previous results, in that it does not restrict the disturbance to be a member of a single statistically defined class such as white noise, etc. The disturbances treated here contain the entire class of disturbances most commonly occurring in practice. This is true because in modern control design the maximum magnitude of the controlling input is sized such that most of the time the disturbances are considered controllable. Thus, the most general class of commonly occurring disturbances of interest to the control designer are those which are uncontrollable for a short duration, followed by long controllable portions.

The success of the technique used in this report lies in assuming that the disturbance is initially known in advance. In this way, a single control function is derived that will yield optimal trajectories for as large a class of disturbances as mathematically possible. This is the best that can theoretically be achieved. Terms contained in the derived control function yield those statistical parameters which are actually required to obtain optimal control. Once the basic form of the necessary statistical parameters has been obtained, the designer can estimate or measure what these parameters are in the actual system for practical optimal control. Experimental results verifying this fact are presented here.

14. KEY WORDS

control systems	optimal control
random disturbances	time optimal control